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# Some applications for subordination principle

Kazuo Kuroki and Shigeyoshi Owa

## Abstract

By considering some subordinations for a more general linear transformation, an extension of the Briot-Bouquet differential subordination relations given by S. S. Miller and P. T. Mocanu (Pure and Applied Mathematics **225**, Marcel Dekker, 2000) for certain linear transformations are discussed.

## 1 Introduction

Let  $\mathcal{H}$  denote the class of functions  $f(z)$  which are analytic in the open unit disk  $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ . For a positive integer  $n$  and a complex number  $a$ , let  $\mathcal{H}[a, n]$  be the class of functions  $f(z) \in \mathcal{H}$  of the form

$$f(z) = a + \sum_{k=n}^{\infty} a_k z^k.$$

Also, let  $\mathcal{A}_n$  denote the class of functions  $f(z) \in \mathcal{H}$  of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

with  $\mathcal{A}_1 = \mathcal{A}$ . If  $f(z) \in \mathcal{A}$  satisfies the following inequality

$$\operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U})$$

for some real number  $\alpha$  with  $0 \leq \alpha < 1$ , then  $f(z)$  is said to be starlike of order  $\alpha$  in  $\mathbb{U}$ . This class is denoted by  $\mathcal{S}^*(\alpha)$ . Similarly, we say that  $f(z)$  belongs to the class  $\mathcal{K}(\alpha)$  of convex functions of order  $\alpha$  in  $\mathbb{U}$  if  $f(z) \in \mathcal{A}$  satisfies the following inequality

$$\operatorname{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{U})$$

for some real number  $\alpha$  with  $0 \leq \alpha < 1$ .

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For some real numbers  $A$  and  $B$  with  $-1 \leq B < A \leq 1$ , Janowski [1] has investigated the following linear transformation

$$p(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U})$$

which is analytic and univalent in  $\mathbb{U}$ . This function  $p(z)$  is called the Janowski function. Moreover, as a generalization of the Janowski functions, Kuroki and Owa [2] have discussed the Janowski functions for some complex parameters  $A$  and  $B$  which satisfy

$$(1.1) \quad A \neq B, \quad |B| \leq 1 \quad \text{and} \quad |A - B| + |A + B| \leq 2.$$

Note that the Janowski function defined by the conditions (1.1) is analytic and univalent in  $\mathbb{U}$  and has a positive real part in  $\mathbb{U}$  (see [2]).

We next introduce the familiar principle of differential subordinations between analytic functions. Let  $p(z)$  and  $q(z)$  be members of the class  $\mathcal{H}$ . Then the function  $p(z)$  is said to be subordinate to  $q(z)$  in  $\mathbb{U}$ , written by

$$(1.2) \quad p(z) \prec q(z) \quad (z \in \mathbb{U}),$$

if there exists a function  $w(z)$  which is analytic in  $\mathbb{U}$  with  $w(0) = 0$  and  $|w(z)| < 1$  ( $z \in \mathbb{U}$ ), and such that  $p(z) = q(w(z))$  ( $z \in \mathbb{U}$ ). From the definition of the subordinations, it is easy to show that the subordination (1.2) implies that

$$(1.3) \quad p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset q(\mathbb{U}).$$

In particular, if  $q(z)$  is univalent in  $\mathbb{U}$ , then the subordination (1.2) is equivalent to the condition (1.3).

Miller and Mocanu [4] developed the definitive result concerning the Briot-Bouquet differential subordinations as follows.

**Lemma 1.1** *Let  $n$  be a positive integer, and let  $\beta$  and  $\gamma$  be complex numbers with  $\beta \neq 0$ . Also, let  $h(z)$  be convex and univalent in  $\mathbb{U}$  with  $h(0) = a$ , and suppose that*

$$(1.4) \quad \operatorname{Re}(\beta h(z) + \gamma) > 0 \quad (z \in \mathbb{U})$$

*with  $\operatorname{Re}(\beta a + \gamma) > 0$ . If  $p(z) \in \mathcal{H}[a, n]$  with  $p(z) \not\equiv a$  satisfies the differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \quad (z \in \mathbb{U}),$$

*then  $p(z) \prec q(z) \prec h(z)$  ( $z \in \mathbb{U}$ ), where  $q(z)$  with  $q(0) = a$  is the univalent solution of the differential equation*

$$q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = h(z) \quad (z \in \mathbb{U}).$$

As applications of Lemma 1.1, Miller and Mocanu [4] derived some subordination relation for certain linear transformations.

**Lemma 1.2** *Let  $n$  be a positive integer. Also, let  $\beta$ ,  $\gamma$  and  $A$  be complex numbers with  $\operatorname{Re}(\beta + \gamma) > 0$ , and let  $B$  be a real number with  $-1 \leq B \leq 0$ . If  $\beta$ ,  $\gamma$ ,  $A$  and  $B$  satisfy either*

$$\operatorname{Re}(\beta(1 + AB) + \gamma(1 + B^2)) \geq |\beta A + \bar{\beta}B + 2B\operatorname{Re}\gamma| \quad (-1 < B \leq 0)$$

or

$$\beta(1 + A) > 0 \quad \text{and} \quad \operatorname{Re}(\beta(1 + A) + 2\gamma) \geq 0 \quad (B = -1),$$

then  $p(z) \in \mathcal{H}[1, n]$  with  $p(z) \not\equiv 1$  satisfies the following subordination relation

$$(1.5) \quad p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + Az}{1 + Bz} \quad \text{implies} \quad p(z) \prec q(z) \prec \frac{1 + Az}{1 + Bz}$$

for  $z \in \mathbb{U}$ , where  $q(z)$  with  $q(0) = a$  is the univalent solution of the differential equation

$$(1.6) \quad q(z) + \frac{nzq'(z)}{\beta q(z) + \gamma} = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}).$$

In the present paper, applying the theory of subordinations, we will try to determine the best conditions for complex numbers  $\beta$ ,  $\gamma$ ,  $A$  and  $B$  to satisfy the condition (1.4) as

$$h(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U})$$

in Lemma 1.1, and deduce an extension of Lemma 1.2.

## 2 Some subordinations for certain linear transformations

By using the method of a certain generalization of the Janowski functions given by Kuroki and Owa [2], we first consider a certain subordination for a more general linear transformation.

**Theorem 2.1** *Let  $a$ ,  $A$ ,  $B$ ,  $C$  and  $D$  be complex numbers with  $A \neq aB$  and  $C \neq aD$ . If  $a$ ,  $A$ ,  $B$ ,  $C$  and  $D$  satisfy  $|B| \leq 1$ ,  $|D| \leq 1$  and*

$$(2.1) \quad |A - aB| + |AD - BC| \leq |C - aD|,$$

then

$$(2.2) \quad \frac{a + Az}{1 + Bz} \prec \frac{a + Cz}{1 + Dz} \quad (z \in \mathbb{U}).$$

*Proof.* From  $A \neq aB$  and the inequality (2.1), it is clear that

$$(2.3) \quad |C - aD| - |AD - BC| > 0.$$

If we define the function  $w(z)$  by

$$(2.4) \quad w(z) = \frac{(A - aB)z}{C - aD - (AD - BC)z} \quad (z \in \mathbb{U}),$$

then from the inequality (2.3),  $w(z)$  is analytic in  $\mathbb{U}$  with  $w(0) = 0$ , and that

$$\frac{a + Az}{1 + Bz} = \frac{a + Cw(z)}{1 + Dw(z)} \quad (z \in \mathbb{U}).$$

Further, noting the inequality (2.3), a simple calculation yields

$$\left| w(z) - \frac{(A - aB)(\overline{AD - BC})}{|C - aD|^2 - |AD - BC|^2} \right| < \frac{|A - aB||C - aD|}{|C - aD|^2 - |AD - BC|^2} \quad (z \in \mathbb{U}).$$

Since the inequality (2.1) shows that

$$\frac{|A - aB|}{|C - aD| - |AD - BC|} \leq 1,$$

we see that  $w(z)$  defined by (2.4) satisfies  $|w(z)| < 1$  ( $z \in \mathbb{U}$ ). Therefore, from the definition of the subordinations, we conclude that the subordination (2.2) holds, which completes the proof of Theorem 2.1.  $\square$

In particular, letting

$$A = b, \quad C = \bar{a}e^{i\theta} \quad \text{and} \quad D = -e^{i\theta}$$

for a complex number  $a$  with  $\operatorname{Re} a > 0$  and for some  $\theta$  with  $0 \leq \theta < 2\pi$  in Theorem 2.1, we find the following assertion.

**Corollary 2.2** *Let  $a$  be a complex number with  $\operatorname{Re} a > 0$ . For some complex numbers  $a, b$  and  $B$  with*

$$b \neq aB, \quad |B| \leq 1 \quad \text{and} \quad |b - aB| + |b + \bar{a}B| \leq 2\operatorname{Re} a,$$

*we have*

$$\frac{a + bz}{1 + Bz} \prec \frac{a + \bar{a}e^{i\theta}z}{1 - e^{i\theta}z} \quad (z \in \mathbb{U}),$$

*where  $0 \leq \theta < 2\pi$ . This subordination means the following inequality*

$$\operatorname{Re} \left( \frac{a + bz}{1 + Bz} \right) > 0 \quad (z \in \mathbb{U}).$$

**Remark 2.3** Taking  $a = 1$  and  $b = A$  in Corollary 2.2, we find the conditions in (1.1) as the conditions for complex numbers  $A$  and  $B$  to satisfy

$$\operatorname{Re} \left( \frac{1 + Az}{1 + Bz} \right) > 0 \quad (z \in \mathbb{U}).$$

### 3 The Briot-Bouquet differential subordinations for certain linear transformations

By using the discussion in the previous section, and applying Lemma 1.1, we deduce an improvement of Lemma 1.2 bellow.

**Theorem 3.1** *Let  $n$  be a positive integer, and let  $\beta, \gamma, A$  and  $B$  be complex numbers with  $\operatorname{Re}(\beta + \gamma) > 0$ ,  $A \neq B$  and  $|B| \leq 1$ . If  $\beta, \gamma, A$  and  $B$  satisfy*

$$|\beta(A - B)| + |\beta(A - B) + 2B\operatorname{Re}(\beta + \gamma)| \leq 2\operatorname{Re}(\beta + \gamma),$$

*then  $p(z) \in \mathcal{H}[1, n]$  with  $p(z) \not\equiv 1$  satisfies the subordination relation (1.5), where  $q(z)$  with  $q(0) = 1$  is the solution of the differential equation (1.6).*

*Proof.* If we let

$$a = \beta + \gamma, \quad b = \beta A + \gamma B \quad \text{and} \quad h(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}),$$

then, a simple check gives us that

$$b - aB = (\beta A + \gamma B) - (\beta + \gamma)B = \beta(A - B) \neq 0$$

and

$$\begin{aligned} 2\operatorname{Re} a - (|b - aB| + |b + \bar{a}B|) &= 2\operatorname{Re} a - (|b - aB| + |b - aB + 2B\operatorname{Re} a|) \\ &= 2\operatorname{Re}(\beta + \gamma) - (|\beta(A - B)| + |\beta(A - B) + 2B\operatorname{Re}(\beta + \gamma)|) \geq 0. \end{aligned}$$

Hence by Corollary 2.2, it is easy to see that

$$\operatorname{Re}(\beta h(z) + \gamma) = \operatorname{Re}\left(\frac{\beta + \gamma + (\beta A + \gamma B)z}{1 + Bz}\right) = \operatorname{Re}\left(\frac{a + bz}{1 + Bz}\right) > 0 \quad (z \in \mathbb{U}).$$

Therefore, since the conditions of Lemma 1.1 are satisfied, we conclude the assertion of Theorem 3.1.  $\square$

By taking  $\beta = 1$ ,  $\gamma = 0$  and  $n = 1$  in Theorem 3.1, and letting

$$p(z) = \frac{zf'(z)}{f(z)} \quad (z \in \mathbb{U})$$

for  $f(z) \in \mathcal{A}$ , we obtain the following subordination implication.

**Corollary 3.2** *If  $f(z) \in \mathcal{A}$  satisfies*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U})$$

*for some complex numbers  $A$  and  $B$  which satisfy the conditions in (1.1), then*

$$\frac{zf'(z)}{f(z)} \prec \begin{cases} \frac{Az}{(1 + B)\{1 - (1 + Bz)^{-\frac{A}{B}}\}} & (A \neq 0, B \neq 0) \\ \frac{Bz}{(1 + Bz)\log(1 + Bz)} & (A = 0, B \neq 0) \\ \frac{Aze^{Az}}{e^{Az} - 1} & (A \neq 0, B = 0) \end{cases}$$

for  $z \in \mathbb{U}$ .

Moreover, let us consider the case that

$$A = 1 - 2\alpha \quad (0 \leq \alpha < 1) \quad \text{and} \quad B = -1$$

in Corollary 3.2. Then, from the definition of the subordinations, we find the implication that if  $f(z) \in \mathcal{K}(\alpha)$ , then  $f(z) \in \mathcal{S}^*(\beta)$ , where

$$\beta = \beta(\alpha) = \begin{cases} \frac{1 - 2\alpha}{2^{2-2\alpha}(1 - 2^{2\alpha-1})} & \left(\alpha \neq \frac{1}{2}\right) \\ \frac{1}{2 \log 2} & \left(\alpha = \frac{1}{2}\right) \end{cases}$$

for each real number  $\alpha$  with  $0 \leq \alpha < 1$ . This relationship for convex and starlike functions was proven by MacGregor [3].

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